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ON SOME PROPERTIES OF KOROBOV POLYNOMIALS

We represent Korobov polynomials as paradeterminants of triangular matrices and prove some of their properties.

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INTRODUCTION

Korobov in [2] introduces polynomials of a special form, which are discrete analogs of Bernoulli polynomials. These polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula [3]. Therefore, it is topical to conduct further research of their properties.

1 OVERVIEW ON TRIANGULAR MATRICES AND THEIR PARADETERMINANTS

Definition 1 ([4]). A triangular table of numbers from some field $K$

$$A = 
\begin{pmatrix}
a_{11} & a_{22} \\
a_{21} & a_{22} \\
:: & :: \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}_n
$$

is called a triangular matrix, and the number $n$ — its order.

Note, that in our understanding a triangular matrix is not a matrix in its usual sense, it is a triangular but not rectangular table of numbers.

To every elements $a_{ij}$ of the matrix (1) we correspond the $(i - j + 1)$ elements $a_{ik}$, $k = j, \ldots, i$, which are called the derived elements of the matrix, generated by the key element $a_{ij}$.

The product of all derived elements generated by the element $a_{ij}$ is denoted by $\{a_{ij}\}$ and is called the factorial product of the key element $a_{ij}$, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$
Definition 2. The paradeterminant and the parapermanent of the triangular matrix
\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \]
are, respectively, the functions
\[ \text{ddet}(A) = \sum_{r=1}^{n} \sum_{p_{1} + \ldots + p_{r} = n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_{1}+\ldots+p_{s}p_{1}+\ldots+p_{s-1}+1}\}, \]
\[ \text{pper}(A) = \sum_{r=1}^{n} \sum_{p_{1} + \ldots + p_{r} = n} \prod_{s=1}^{r} \{a_{p_{1}+\ldots+p_{s}p_{1}+\ldots+p_{s-1}+1}\}. \]

To every element \(a_{ij}\) of the triangular matrix (1) we correspond the triangular matrix with this element in the bottom left corner, which is called a corner of the triangular matrix and denoted by \(R_{ij}(A)\). It is obvious that the corner \(R_{ij}(A)\) is the triangular matrix of the \((i - j + 1)\)-th order. The corner \(R_{ij}(A)\) comprises only those elements \(a_{rs}\) of the triangular matrix (1), the indexes of which satisfy the relations \(j \leq s \leq r \leq i\).

The parafunctions of triangular matrices can be decomposed by the elements of their last row:
\[ \text{ddet} (A) = \sum_{s=1}^{n} (-1)^{n+s} \{a_{ns}\} \cdot \text{ddet} (R_{s-1,1}), \]
\[ \text{pper} (A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot \text{pper}(R_{s-1,1}). \]

Proposition. The following is true:
\[ \text{pper} (A) := \begin{pmatrix} a_{1} \\\n a_{2} \\\n \vdots \\\n a_{n} \end{pmatrix} a_{n-1} \cdots a_{1} \begin{pmatrix} a_{0} \\\n a_{0} \cdots a_{2} \vdots \vdots \vdots a_{m} \\ 0 \end{pmatrix} = \sum_{\lambda_{1}+2\lambda_{2}+\ldots+n\lambda_{n}=m} \frac{k!}{\lambda_{1}!\lambda_{2}!\cdots\lambda_{n}!} a_{1}^{\lambda_{1}}a_{2}^{\lambda_{2}}\cdots a_{n}^{\lambda_{n}}, \]
and
\[ \text{ddet} (A) := \begin{pmatrix} a_{1} \\ a_{2} \\\n \vdots \\ a_{n} \end{pmatrix} a_{n-1} \cdots a_{1} \begin{pmatrix} a_{0} \\\n a_{0} \cdots a_{2} \vdots \vdots \vdots a_{m} \\ 0 \end{pmatrix} = \sum_{\lambda_{1}+2\lambda_{2}+\ldots+n\lambda_{n}=m} \frac{(-1)^{n-k}k!}{\lambda_{1}!\lambda_{2}!\cdots\lambda_{n}!} a_{1}^{k}a_{2}^{\lambda_{2}+\ldots+\lambda_{n}}\cdots a_{n}^{\lambda_{n}}, \]
where \( k = \lambda_1 + \lambda_2 + \ldots + \lambda_n \).

For more detailed information on triangular matrices and their paraderivants, the reader is referred to [4], [5].

2 Korobov Polynomials and Paraderivants

In [2] the Korobov numbers \( P_n \) and polynomials \( P_n(x) \) are defined by the equalities

\[
P_0 = 1, \quad \left( \begin{array}{l} p \\ 1 \end{array} \right) P_n + \ldots + \left( \begin{array}{l} p \\ n+1 \end{array} \right) P_0 = 0, \quad n \geq 1; \quad (3)
\]

\[
P_0(x) = 1, \quad P_n(x) = P_0 \left( \frac{x}{n} \right) + \ldots + P_{n-1} \left( \frac{x}{1} \right) + P_n, \quad n \geq 1.
\]

We shall write the Korobov numbers as the paraderivant of the triangular matrix.

**Theorem 1.** The following is true

\[
P_n = (-1)^n \begin{vmatrix} \frac{p-1}{n+1} & \frac{p-1}{n} & \ldots & \frac{p-1}{2} \\ \frac{p-2}{n+1} & \ldots & \frac{p-2}{2} \\ \vdots & \ldots & \vdots \\ \frac{p-n}{n+1} & \frac{p-n+1}{n} & \ldots & \frac{p-n}{2} \end{vmatrix}_n.
\]

**Proof.** Let us divide the second equality (3) by \( \left( \begin{array}{l} p \\ 1 \end{array} \right) \), and we get the recurrence equality

\[
P_n + a_1 P_{n-1} + a_2 P_{n-2} + \ldots + a_{n-1} P_1 + a_n P_0 = 0,
\]

where

\[
a_i = \frac{(p-1)^i}{(i+1)!}.
\]

The last equality, according to [1], has the solution

\[
P_n = (-1)^n \begin{vmatrix} a_1 & a_1 & \ldots & a_1 \\ \frac{a_1}{n!} & \frac{a_1}{n} & \ldots & \frac{a_1}{2} \\ \vdots & \ldots & \vdots & \vdots \\ \frac{a_{n-1}}{n!} & \frac{a_{n-1}}{n} & \ldots & a_1 \end{vmatrix}_n.
\]

That is why, in virtue of the equality

\[
\frac{a_i}{a_{i-1}} = \frac{p-i}{i+1},
\]

the equality (4) is true.

It should be noted that due to the connection between the paraderivants of triangular matrices and the parapermanents of some triangular matrices, the Korobov numbers can also be written as the parapermanent of a triangular matrix.

By now there are several presentations of some algebraic objects as partition polynomials (e.g., Waring’s formula presenting power sums through elementary symmetric polynomials). The following theorem obviously presents the Korobov numbers with the help of the partition polynomials.
Theorem 2. The following is true:

\[ P_n = \sum_{\lambda_1 + \ldots + n\lambda_n = n} (-1)^k \frac{k!}{\lambda_1! \lambda_2! \ldots \lambda_n! (n+1)! \lambda_n!} (p-1)^k (p-2)^{k-\lambda_1} \ldots (p-n)^{\lambda_n}, \]

\[ n = 1, 2, \ldots. \]

Proof. Considering the equality (5) and the identity (2), after some simplifications, we get the presentation of the Korobov numbers as partition polynomials (6).

References


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